3.4) $(x, y) = y^T G x$

En cada caso tengo que probar que se cumpla:

1. $(x+y, z) = (x, z) + (y, z)$
2. $(\lambda x, y) = \lambda (x, y)$
3. $(x, y) = \overline{(y, x)}$
4. $\forall x \neq 0 \rightarrow (x, x) > 0$

a) $\underset{S_1}{\cdot} = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{G_1}, \underbrace{\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}}_{G_2}, \underbrace{\begin{bmatrix} 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix}}_{G_3}, \underbrace{\begin{bmatrix} 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 \end{bmatrix}}_{G_4} \right\}$

Pruebo para cada G_i ~~...~~ con $i = 1, \dots, 4$ que se cumplen las propiedades ①, ②, ③ y ④.

Para G_1:

① $\overline{(x+y, z)} = z^T . G_1 . (x+y) = z^T (G_1 . x + G_1 . y) = z^T G_1 . x + z^T . G_1 . y = \underline{(x,z)+(y,z)}$ ✓

$\boxed{\text{ESTA VERIF. VALE PARA TODOS LOS } G_i, i = 1, \dots, 4.}$

② $\overline{(\lambda x, y)} = y^T . G_1 (\lambda x) = \lambda . y^T . G_1 x = \underline{\lambda . (x, y)}$ ✓

$\boxed{\text{ESTA VERIF. VALE } \forall G_i, i = 1, \dots, 4.}$

③ $\overline{(x,y)} = y^T . G_1 . x = y^T . \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . x \Rightarrow$ Como son vect $\in \mathbb{R}^2$:

$\rightarrow = [y_1 \; y_2] . \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [y_1 \; y_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{y_1 . x_1 + y_2 . x_2}_{\triangle}$

$\underline{(\overline{y,x})} = \overline{x^T} . \overline{G_1} . \overline{y} \Rightarrow$ Como son nros. reales $\longrightarrow$

$\rightarrow = x^T . G_1 . y = [x_1 \; x_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [x_1 \; x_2] . \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + x_2 y_2 \rightarrow$

$\rightarrow$ Ordenando $\rightarrow = y_1 x_1 + y_2 x_2 = \triangle$ ✓ $\rightarrow \overline{(x,y)} = \underline{(\overline{y,x})}$

④ $\overline{(x,x)} = x^T G_1 . x = [x_1 \;\; x_2]\, G_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \;\; x_2] . \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow$

$\rightarrow = [x_1 \;\; x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 > 0 \quad \forall x \neq 0$ ✓

Para G_2

③ $\overline{(x,y)} = y^T . G_2 . x = [y_1 \;\; y_2] . \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[y_1 + \frac{y_2}{2} \;\; \frac{y_1}{2} + y_2 \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow$

$\rightarrow = \underbrace{\left(y_1 + \frac{y_2}{2}\right) x_1 + \left(\frac{y_1}{2} + y_2\right) x_2}_{\triangle}$

$\overline{(y,x)} = \overline{x^T} . \overline{G_2} . \overline{y} = x^T . G_2 . y = [x_1 \;\; x_2] . \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \longrightarrow$

$\rightarrow = \left[x_1 + \frac{x_2}{2} \;\; \frac{x_1}{2} + x_2 \right] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left(x_1 + \frac{x_2}{2}\right) y_1 + \left(\frac{x_1}{2} + x_2\right) y_2 \longrightarrow$

$\rightarrow = x_1 y_1 + \frac{x_2}{2} y_1 + \frac{x_1}{2} y_2 + x_2 y_2 = \left(y_1 + \frac{y_2}{2}\right) x_1 + \left(\frac{y_1}{2} + y_2\right) x_2 = \triangle$ ✓

$\rightarrow \overline{(x,y)} = (y,x)$

④ $\overline{(x,x)} = x^T . G_2 . x = [x_1 \;\; x_2] . \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow$

$\rightarrow = \left[x_1 + \frac{x_2}{2} \;\; \frac{x_1}{2} + x_2 \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(x_1 + \frac{x_2}{2}\right) x_1 + \left(\frac{x_1}{2} + x_2\right) x_2 \longrightarrow$

$\rightarrow = x_1^2 + \frac{x_2}{2} x_1 + \frac{x_1}{2} x_2 + x_2^2 = x_1^2 + x_1 x_2 + x_2^2$

$\to x_1^2 + x_1 x_2 + x_2^2 \geq x_1^2 - |x_1 x_2| + x_2^2 = x_1^2 - (|x_1|.|x_2|) + x_2^2$

$\to x_1^2 + x_1 x_2 + x_2^2 = \left(x_1 + \frac{1}{2}\cancel{x_2}\right)^2 + \left(\frac{3}{4}\cancel{x_2}^2\right) > 0 \quad \forall\, x \neq 0$ ✓

Para G_3

3 $\overline{(x,y)} = y^T . G_3 . x = [y_1 \ \ y_2] . \begin{bmatrix} 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[y_1 + \frac{\sqrt{2}}{2} y_2 \quad \frac{\sqrt{2}}{2} y_1 + y_2 \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to$

$\to = \underbrace{\left(y_1 + \frac{\sqrt{2}}{2} y_2\right) x_1 + \left(\frac{\sqrt{2}}{2} y_1 + y_2\right) x_2}_{A}$

$\overline{(y,x)} = \overline{x^T} . \overline{G_3} . \overline{y} = x^T . G_3 . y = [x_1 \ \ x_2] . \begin{bmatrix} 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \to$

$\to = \left[x_1 + x_2 \frac{\sqrt{2}}{2} \quad x_1 \frac{\sqrt{2}}{2} + x_2 \right] . \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left(x_1 + \frac{\sqrt{2}}{2} x_2\right) y_1 + \left(\frac{\sqrt{2}}{2} x_1 + x_2\right) y_2 \to$

$\to = x_1 y_1 + \frac{\sqrt{2}}{2} x_2 y_1 + \frac{\sqrt{2}}{2} x_1 y_2 + x_2 y_2 = \left(y_1 + \frac{\sqrt{2}}{2} y_2\right) x_1 + \left(\frac{\sqrt{2}}{2} y_1 + y_2\right) x_2 = A$ ✓

$\to \overline{(x,y)} = \overline{(y,x)}$

④ $\overline{(x,x)} = x^T \cdot G_3 \cdot x = [x_1 \;\; x_2] \cdot \begin{bmatrix} 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + \frac{\sqrt{2}}{2} x_2 & \frac{\sqrt{2}}{2} x_1 + x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow$

$\rightarrow = \left(x_1 + \frac{\sqrt{2}}{2} x_2\right) x_1 + \left(\frac{\sqrt{2}}{2} x_1 + x_2\right) x_2 = x_1^2 + \sqrt{2}\, x_1 x_2 + x_2^2 \longrightarrow$

$\rightarrow = \left(x + \frac{\sqrt{2}}{2} x_2\right)^2 + \frac{1}{2} x_2^2 \; \underline{> 0} \quad \forall\, x \neq 0$ ✓

Para G_4

③ $\overline{(x,y)} = y^T \cdot G_4 \cdot x = [y_1 \;\; y_2] \cdot \begin{bmatrix} 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 + \frac{\sqrt{3}}{2} y_2 & \frac{\sqrt{3}}{2} y_1 + y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow$

$\rightarrow = \underbrace{\left(y_1 + \frac{\sqrt{3}}{2} y_2\right) x_1 + \left(\frac{\sqrt{3}}{2} y_1 + y_2\right) x_2}_{\Delta}$

$\underline{\overline{(y,x)}} = \overline{x^T} \cdot \overline{G_4} \cdot \overline{y} = x^T \cdot G_4 \cdot y = [x_1 \;\; x_2] \cdot \begin{bmatrix} 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \longrightarrow$

$\rightarrow = \begin{bmatrix} x_1 + \frac{\sqrt{3}}{2} x_2 & \frac{\sqrt{3}}{2} x_1 + x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left(x_1 + \frac{\sqrt{3}}{2} x_2\right) y_1 + \left(\frac{\sqrt{3}}{2} x_1 + x_2\right) y_2 \longrightarrow$

$\rightarrow = x_1 y_1 + \frac{\sqrt{3}}{2} x_2 y_1 + \frac{\sqrt{3}}{2} x_1 y_2 + x_2 y_2 = \left(y_1 + \frac{\sqrt{3}}{2} y_2\right) x_1 + \left(\frac{\sqrt{3}}{2} y_1 + y_2\right) x_2 = \Delta$ ✓

$\rightarrow \overline{(x,y)} = \underline{\overline{(y,x)}}$

④ $\overline{(x,x)} = x^T . G_4 . x = [x_1 \;\; x_2]\begin{bmatrix}1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}x_1+\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}x_1+x_2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \to$

$\to = \left(x_1+\frac{\sqrt{3}}{2}x_2\right)x_1 + \left(\frac{\sqrt{3}}{2}x_1+x_2\right)x_2 = x_1^2+\sqrt{3}\,x_1x_2+x_2^2 \longrightarrow$

$\to = \left(x_1+\frac{\sqrt{3}}{2}x_2\right)^2 + \frac{1}{4}x_2^2 > 0 \quad \forall\, x \neq 0$ ✓

Por lo tanto $(x,y) = y^T G x$ define um prod. interno em $\mathbb{R}^2$

b) $S_2 = \left\{ \underbrace{\begin{bmatrix}1 & \cos\theta \\ \cos\theta & 1\end{bmatrix}}_{G} : \theta \in (0,\pi) \right\}$

① $\overline{(x+y,z)} = z^T G(x+y) = z^T G x + z^T G y = (x,z)+(y,z)$ ✓

② $\overline{(\lambda x,y)} = y^T G(\lambda x) = \lambda\, y^T G x = \lambda(x,y)$ ✓

③ $\overline{(x,y)} = y^T . G . x = [y_1 \;\; y_2]\begin{bmatrix}1 & \cos\theta \\ \cos\theta & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = [y_1+y_2\cos\theta \quad y_1\cos\theta+y_2]\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \to$

$\to = \underbrace{(y_1+y_2\cos\theta)x_1 + (y_1\cos\theta+y_2)x_2}_{\Delta}$

$\overline{(y,x)} = \bar{x}^T . \bar{G} . \bar{y} = x^T . G . y = [x_1 \;\; x_2]\begin{bmatrix}1 & \cos\theta \\ \cos\theta & 1\end{bmatrix}\begin{bmatrix}y_1 \\ y_2\end{bmatrix} \longrightarrow$

$\to = (x_1+x_2\cos\theta)y_1 + (x_1\cos\theta+x_2)y_2 = x_1y_1+x_2\cos\theta\, y_1 + x_1\cos\theta\, y_2 + x_2y_2 +$

$\to = (y_1+y_2\cos\theta)x_1 + (y_1\cos\theta+y_2)x_2 = \Delta$ ✓

$\to \overline{(x,y)} = \overline{(y,x)}$

$④\ \overline{(x,x)} = x^T . G . x = [x_1\ x_2] \begin{bmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow$

$\rightarrow = (x_1 + x_2\cos\theta)x_1 + (x_1\cos\theta + x_2)x_2 = x_1^2 + 2x_1x_2\cos\theta + x_2^2 \rightarrow$

$\rightarrow = (x_1 . \cos\theta + x_2)^2 + \operatorname{sen}^2\theta . x_1^2 \underline{> 0} \quad \forall x \neq 0 .$ ✓

Por lo tanto $(x,y) = y^T G x$ define un PI en $\mathbb{R}^2$. ✓

c) $S_3 = \left\{ \underbrace{\begin{bmatrix} l_1^2 & l_1 l_2 \cos\theta \\ l_1 l_2\cos\theta & l_2^2 \end{bmatrix}}_{G} : \theta \in (0,\pi), l_1 > 0, l_2 > 0 \right\}$

$①\ \overline{(x+y,z)} = z^T G (x+y) = z^T G x + z^T G y = \underline{(x,z) + (y,z)}$ ✓

$②\ \overline{(\lambda x, y)} = y^T G (\lambda x) = \lambda\ y^T G x = \underline{\lambda . (x,y)}$ ✓

$③\ \overline{(x,y)} = y^T G x = [y_1\ y_2] \begin{bmatrix} l_1^2 & l_1 l_2\cos\theta \\ l_1l_2\cos\theta & l_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow$

$\rightarrow = \underbrace{(y_1 l_1^2 + y_2 l_1 l_2 \cos\theta)x_1 + (y_1 l_1 l_2 \cos\theta + y_2 l_2^2)x_2}_{\Delta}$

$\underline{(\overline{y,x})} = \overline{x^T} . \overline{G} . \overline{y} = x^T . G . y = [x_1\ x_2] \begin{bmatrix} l_1^2 & l_1 l_2 \cos\theta \\ l_1 l_2\cos\theta & l_2^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \longrightarrow$

$\rightarrow = (x_1 l_1^2 + x_2 l_1 l_2 \cos\theta)y_1 + (x_1 l_1 l_2 \cos\theta + x_2 l_2^2)y_2 \longrightarrow$

$\rightarrow = (y_1 l_1^2 + y_2 l_1 l_2 \cos\theta)x_1 + (y_1 l_1 l_2\cos\theta + y_2 l_2^2) = \Delta$ ✓

$\rightarrow\ \overline{(x,y)} = \underline{(\overline{y,x})}$

④ $\overline{(x,x)} = x^T G x = [x_1\ x_2]\begin{bmatrix} l_1^2 & l_1 l_2 \cos\theta \\ l_1 l_2 \cos\theta & l_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow$

$\rightarrow = (x_1 l_1^2 + x_2 l_1 l_2 \cos\theta)x_1 + (x_1 l_1 l_2 \cos\theta + x_2 l_2^2)x_2 \longrightarrow$

$\rightarrow = x_1^2 l_1^2 + 2x_1x_2 l_1 l_2 \cos\theta + x_2^2 l_2^2 = (x_1 l_1 \cos\theta + x_2 l_2)^2 + x_1^2 l_1^2 \operatorname{sen}^2\theta > 0\ \forall x \neq 0$ ✓

ya que $l_1, l_2 > 0$
por enunciado.

$\boxed{\text{Por lo tanto } (x,y) = y^T G x \text{ define un PI en } \mathbb{R}^2}$

d) Se puede probar que $S_3 = S_4$ (con doble inclusión) por lo que, como en c) vimos que S_3 cumplía todo, S_4 también lo hará.